The contact process: recent results on finite-volume phase transitions
Mini-course overview

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Overview of the course

Goals of the course:

- Give an overview of interacting particle systems
- Present some of the classical theory of the contact process
- Show recent results of the contact process on various finite graphs (deterministic and random, static and dynamic)

Outline of lectures

- **Lecture 1:** Introduction to interacting particle systems
- **Lecture 2:** Introduction to the contact process
- **Lecture 3:** Contact process on finite graphs (1)
- **Lecture 4:** Contact process on finite graphs (2)

Rest of these slides

- What are **interacting particle systems**?
- What is the **contact process**?
- What do you mean by **phase transition**?
- What do you mean by **finite-volume phase transition**?

Interacting particle systems

Sub-field of stochastic processes

Beginnings: 70's – models in statistical physics, study of processes involving many random walks

No firm definition, but some key features are:

- large number of entities, typically arranged in some graph or geometry (so that there's a notion of "locality"),
- each entity can be in one out of a set of possible states,
- evolution is random, Markovian, with rules involving local rates and updates.

Modelling framework in many fields of science (fine description, to microscopic level)

Interacting particle systems

Some typical directions of investigation:

- Trajectory properties (what does the process do?)
  Focus on threshold phenomena (phase transitions)
- Ergodic properties: what are the equilibria of the system?
  Under which conditions, and how fast, are they attained?
- Scaling limits: take sequence of processes where space and time are rescaled, obtain limiting process

Interacting particle systems

References:

- Liggett, Interacting particle systems, 1985
- Liggett, Stochastic interacting systems, 1999
- Swart, A course in interacting particle systems, 2017

The contact process

Introduced by Ted Harris in 1974

Model for the spread of an infection in a population

$G = (V, E)$ graph, $\lambda > 0$ (infection rate)
At any point in time, each vertex (individual) may be:
healthy (state 0) or infected (state 1)

Rules of dynamics: each infected vertex

- becomes healthy with rate 1;
- transmits the infection to each neighbor with rate λ

The contact process

Very versatile model, basis for:

- models of competition
- biological models with sexual reproduction, maturation
- vegetation models
- gene regulatory networks

Mathematically tractable, relevant for theoretical advances in:

- phase transitions, criticality
- shape theorems
- processes on random environments

Phase transition on $\mathbb{Z}^d$

For the contact process on $\mathbb{Z}^d$ started from infection at the origin:

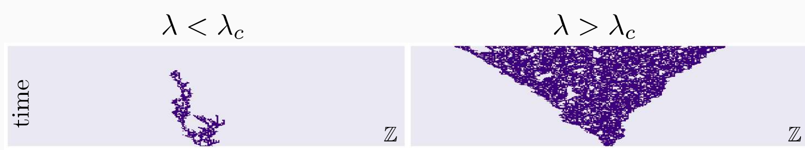
Theorem

[Harris, **Annals of Probability** 1974]

[Bezuidenhout & Grimmett, **Annals of Probability** 1990]

There exists $\lambda_c = \lambda_c(\mathbb{Z}^d) \in (0, \infty)$ such that

$$\mathbb{P}\,(\text{infection eventually disappears}) \begin{cases} = 1 \text{ if } \lambda \le \lambda_c; \\ < 1 \text{ if } \lambda > \lambda_c. \end{cases}$$



Finite graphs

Fact: If G is a finite graph, then for the contact process on G,

$$\mathbb{P}(\text{infection eventually disappears}) = 1$$

(regardless of λ or the initial configuration)

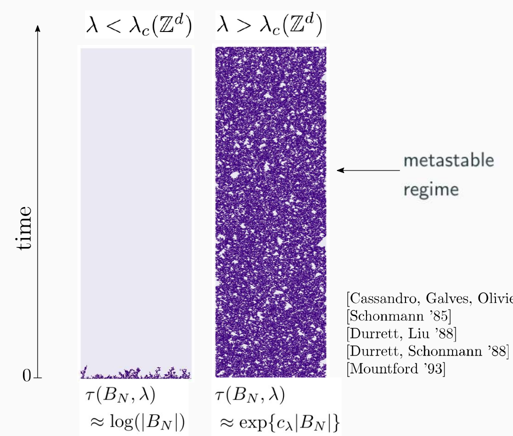
Define the **extinction time of the infection**

$$\tau(G, \lambda) := \min\{t : \text{infection absent at time } t\},$$

for process started from everyone infected

Finite-volume phase transition: boxes of $\mathbb{Z}^d$

Let $B_N := \{1, \dots, N\}^d$, fix $\lambda > 0$, study $\tau(B_N, \lambda)$...



[Cassandro, Galves, Olivieri, Vares '84]
[Schonmann '85]
[Durrett, Liu '88]
[Durrett, Schonmann '88]
[Mountford '93]

Summarizing

Infinite graphs (lattices, regular trees...): phase transition

extinction vs. survival of the infection

Finite graphs: infection eventually disappears. Nevertheless, infinite-volume phase transitions have a counterpart:

quick vs. slow extinction

"Slow" means *very* slow (metastability).

Collaborators

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- Gabriel Leite Baptista da Silva (University of Groningen)
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- Michael Cranston (UC Irvine)
- Roberto Imbuzeiro Oliveira (IMPA)
- Tom Mountford (EPFL)
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