LECTURE NOTES ON APPLICATIONS OF GROTHENDIECK'S INEQUALITY

QUASIRANDOM GRAPHS

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ABSTRACT. In this lecture we cover a result of Conlon and Zhao [CZ17] on the equivalence of quasirandom properties of sparse graphs.

1. QUASIRANDOM GRAPHS

A graph is *d-regular* if each vertex is contained in exactly d edges. For a graph $G = (V, E)$ and subsets $S, T \subseteq V$, denote by $e(S, T)$ the number of edges with an endpoint in both S and T. The adjacency matrix of G is the matrix $A \in \mathbb{R}^{V \times V}$ given by $A_{u,v} = e(\{u\}, \{v\})$.

Definition 1.1. An n-vertex d-regular graph $G = (V, E)$ is *ε-uniform* if for all vertex-subsets $S, T \subseteq V$, we have

$$\left| e(S,T) - \frac{d}{n}|S||T| \right| \leq \varepsilon dn. \tag{1}$$

Denote by $\Delta(G)$ the smallest ε such that G is ε-uniform.

Definition 1.2. A graph G is an (n, d, λ)-graph if it has n vertices, degree d and all the eigenvalues of its adjacency matrix, except the largest, are at most λ in absolute value. Denote by $\lambda(G)$ the smallest λ such that G is an (n, d, λ)-graph.

Lemma 1.3 (Expander mixing lemma)**.** *Let $G = (V, E)$ be an (n, d, λ)-graph. Then, for all vertex-subsets $S, T \subseteq V$, we have*

$$\left| e(S,T) - \frac{d}{n}|S||T| \right| \leq \lambda\sqrt{|S||T|}. \tag{2}$$

In particular, an (n, d, λ)-graph is (λd^{-1})-uniform. A famous result of Chung, Graham and Wilson [CGW89] shows that the converse of Lemma 1.3 holds for dense graphs (in which case $d \geq \Omega(n)$). In particular, for any $\delta > 0$, there is a $C(\delta) > 0$ such any n-vertex d-regular

ε-uniform graph with $d \geq \delta n$ is an (n, d, λ)-graph with $\lambda \leq C(\delta)\varepsilon d$. For sparse graphs, such a converse no longer holds in general. It was shown in [CZ17] that there exist d-regular n-vertex graphs with $d \to \infty$ as $n \to \infty$ that are $o(1)$-uniform, but for which $\lambda \geq \Omega(d)$. It turns out that this situation cannot occur for sparse graphs with a certain amount of symmetry. For a graph $G = (V, E)$, an *automorphism* is a permutation $\pi : V \to V$ such that $\{\pi(u), \pi(v)\} \in E$ if and only if $\{u, v\} \in E$.

Definition 1.4. A graph G is *vertex transitive* if for every pair of vertices u, v, there exists an automorphism π such that $\pi(u) = v$.

Theorem 1.5 (Conlon–Zhao)**.** *Any n-vertex d-regular graph that is vertex transitive and ε-uniform is an (n, d, λ)-graph for $\lambda \leq 4\varepsilon K_G d$, where $K_G \in (1, 2)$ is the Grothendieck constant.*

Bilu and Linial [BL06] proved that if a d-regular graph satisfies the stronger condition that the left-hand side of (1) is at most $\varepsilon d\sqrt{|S||T|}$, then $\lambda \leq C\varepsilon d \log(2/\varepsilon)$ for some absolute constant $C > 0$. Theorem 1.5 gives the stronger conclusion $\lambda \leq 4\varepsilon K_G d$ from the weaker condition (1).

2. A link with Grothendieck's inequality

The proof of Theorem 1.5 uses Grothendieck's inequality, which we now recall.

Theorem 2.1 (Grothendieck's inequality)**.** *There exists an absolute constant $K_G \in (1, 2)$ such that the following holds. For any positive integer n and matrix $B \in \mathbb{R}^{n\times n}$, we have*

$$\|B\|_G \leq K_G \|B\|_{\infty\to 1}. \tag{3}$$

For an n-vertex d-regular graph G with adjacency matrix A, we use two simple propositions. Let J denotes the all-ones matrix.

Proposition 2.2. *Let G be an n-vertex d-regular graph and let A be its adjacency matrix. Let $B = A - \frac{d}{n}J$. Then,*

$$\lambda(G) = \|B\| \tag{4}$$

$$\|B\|_{\infty\to 1} \leq 4dn\Delta(G). \tag{5}$$

The following key lemma allows us to apply Grothendieck's inequality.

Lemma 2.3. *Let G be a vertex-transitive n-vertex d-regular graph and let A be its adjacency matrix. Let $B = A - \frac{d}{n}J$. Then,*

$$n\|B\| \leq \|B\|_G. \tag{6}$$

Proof of Theorem 1.5: Let A be the adjacency matrix of G and let $B = A - \frac{d}{n}J$. Then, by Proposition 2.2 and Lemma 2.3,

$$n\lambda(G) \overset{(4)}{=} n\|B\| \overset{(6)}{\leq} \|B\|_G \overset{(3)}{\leq} K_G\|B\|_{\infty\to 1} \overset{(5)}{\leq} 4dnK_G\Delta(G).$$

□

3. Proof of Lemma 2.3

The following proof of Lemma 2.3, which is even shorter than the original, follows from Grothendieck's factorization lemma.

Lemma 3.1 (Grothendieck)**.** *For any matrix $A \in \mathbb{R}^{n\times n}$, there exist positive unit vectors $u, v \in \mathbb{R}_{>0} \cap S^{n-1}$ such that for any $x, y \in \mathbb{R}^n$,*

$$|\langle Ax, y\rangle| \leq \|A\|_G \, \|x \circ u\|_2 \|y \circ v\|_2. \tag{7}$$

For a permutation $\pi \in S_n$ and $A \in \mathbb{R}^{n\times n}$, let $A^\pi = (A_{\pi(i),\pi(j)})_{i,j=1}^n$. Observe that if A is the adjacency matrix of a graph G and π is an automorphism of G, then $A^\pi = A$.

Proof of Lemma 2.3: Define $C = B/\|B\|_G$, so that $\|C\|_G = 1$. By Lemma 3.1 and the AMGM inequality, there exist positive unit vectors $u, v \in \mathbb{R}_{>0} \cap S^{n-1}$ such that for any $x, y \in \mathbb{R}^n$,

$$|\langle Cx, y\rangle| \leq \|x \circ u\|_2 \|y \circ v\|_2 \leq \frac{1}{2}\big(\|x \circ u\|_2^2 + \|y \circ v\|_2^2\big).$$

Fix $x, y \in \mathbb{R}^n$. Let $\Gamma \leq S_n$ be the group of automorphisms of G and let Γ act on $\mathbb{R}^n$ in the natural way. Since the adjacency matrix A satisfies $A^\pi = A$ for every $\pi \in \Gamma$, it follows that $C^\pi = C$ for every $\pi \in \Gamma$, and therefore,

$$\langle Cx, y\rangle = \langle C^\pi x, y\rangle = \langle C(\pi^{-1}x), (\pi^{-1}y)\rangle.$$

Putting the above two observations together gives

$$\begin{aligned}
|\langle Cx, y\rangle| &= \big|\mathbb{E}_{\pi\in\Gamma}\big[\langle C(\pi^{-1}x), (\pi^{-1}y)\rangle\big]\big| \\
&\leq \frac{1}{2}\mathbb{E}_{\pi\in\Gamma}\big[\|(\pi^{-1}x) \circ u\|_2^2\big] + \frac{1}{2}\mathbb{E}_{\pi\in\Gamma}\big[\|(\pi^{-1}y) \circ v\|_2^2\big] \\
&= \frac{1}{2}\mathbb{E}_{\pi\in\Gamma}\big[\|x \circ (\pi u)\|_2^2\big] + \frac{1}{2}\mathbb{E}_{\pi\in\Gamma}\big[\|y \circ (\pi v)\|_2^2\big].
\end{aligned}$$

Since Γ is transitive and u is a unit vector, the first expectation on the last line equals

$$\mathbb{E}_{\pi\in\Gamma}\Big[\sum_{i=1}^{n} x_i^2 u_{\pi(i)}^2\Big] = \sum_{i=1}^{n} x_i^2 \mathbb{E}_{\pi\in\Gamma}\big[u_{\pi(i)}^2\big] = n^{-1}\|x\|_2^2.$$

Applying the same argument to the second expectation gives

$$|\langle Cx, y\rangle| \leq \frac{1}{2n}\big(\|x\|_2^2 + \|y\|_2^2\big).$$

Let $\lambda = \sqrt{\|y\|/\|x\|}$. Then, for $x' = \lambda x$ and $y' = \lambda^{-1}y$ we get that

$$2|\langle Cx, y\rangle| = 2|\langle Cx', y'\rangle| \leq \frac{1}{n}\big(\|x'\|^2 + \|y'\|^2\big) = \frac{2}{n}\|x\|\|y\|.$$

This shows that $\|C\| \leq 1/n$ and proves the claim. □

4. Exercises

Exercise 4.1. Prove Lemma 1.3 (the Expander mixing lemma).

Exercise 4.2. Prove Proposition 2.2.

Exercise 4.3. Show that in fact equality holds in (5). [Hint: use the fact that both the rows and the columns of $A - \frac{d}{n}J$ sum to zero.]

Exercise 4.4. Show that equality holds in Lemma 2.3.

References

[BL06] Yonatan Bilu and Nathan Linial. Lifts, discrepancy and nearly optimal spectral gap. *Combinatorica*, 26(5):495–519, 2006.

[CGW89] Fan R. K. Chung, Ronald L. Graham, and Richard M. Wilson. Quasi-random graphs. *Combinatorica*, 9(4):345–362, 1989.

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