LECTURE NOTES ON APPLICATIONS OF
GROTHENDIECK’S INEQUALITY
QUASIRANDOM GRAPHS

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Abstract. In this lecture we cover a result of Conlon and Zhao [CZ17] on the equivalence of quasirandom properties of sparse graphs.

1. Quasirandom graphs

A graph is \( d \)-regular if each vertex is contained in exactly \( d \) edges. For a graph \( G = (V, E) \) and subsets \( S, T \subseteq V \), denote by \( e(S, T) \) the number of edges with an endpoint in both \( S \) and \( T \). The adjacency matrix of \( G \) is the matrix \( A \in \mathbb{R}^{V \times V} \) given by \( A_{u,v} = e(\{u\}, \{v\}) \).

**Definition 1.1.** An \( n \)-vertex \( d \)-regular graph \( G = (V, E) \) is \( \varepsilon \)-uniform if for all vertex-subsets \( S, T \subseteq V \), we have

\[
\left| e(S, T) - \frac{d}{n} |S||T| \right| \leq \varepsilon dn.
\]

Denote by \( \Delta(G) \) the smallest \( \varepsilon \) such that \( G \) is \( \varepsilon \)-uniform.

**Definition 1.2.** A graph \( G \) is an \((n, d, \lambda)\)-graph if it has \( n \) vertices, degree \( d \) and all the eigenvalues of its adjacency matrix, except the largest, are at most \( \lambda \) in absolute value. Denote by \( \lambda(G) \) the smallest \( \lambda \) such that \( G \) is an \((n, d, \lambda)\)-graph.

**Lemma 1.3** (Expander mixing lemma). Let \( G = (V, E) \) be an \((n, d, \lambda)\)-graph. Then, for all vertex-subsets \( S, T \subseteq V \), we have

\[
\left| e(S, T) - \frac{d}{n} |S||T| \right| \leq \lambda \sqrt{|S||T|}.
\]

In particular, an \((n, d, \lambda)\)-graph is \((\lambda d^{-1})\)-uniform. A famous result of Chung, Graham and Wilson [CGW89] shows that the converse of Lemma 1.3 holds for dense graphs (in which case \( d \geq \Omega(n) \)). In particular, for any \( \delta > 0 \), there is a \( C(\delta) > 0 \) such any \( n \)-vertex \( d \)-regular
\[ \varepsilon \text{-uniform graph with } d \geq \delta n \text{ is an } (n, d, \lambda) \text{-graph with } \lambda \leq C(\delta)\varepsilon d. \] For sparse graphs, such a converse no longer holds in general. It was shown in [CZ17] that there exist \( d \)-regular \( n \)-vertex graphs with \( d \to \infty \) as \( n \to \infty \) that are \( o(1) \)-uniform, but for which \( \lambda \geq \Omega(d) \). It turns out that this situation cannot occur for sparse graphs with a certain amount of symmetry. For a graph \( G = (V, E) \), an automorphism is a permutation \( \pi : V \to V \) such that \( \{\pi(u), \pi(v)\} \in E \) if and only if \( \{u, v\} \in E \).

**Definition 1.4.** A graph \( G \) is **vertex transitive** if for every pair of vertices \( u, v \), there exists an automorphism \( \pi \) such that \( \pi(u) = v \).

**Theorem 1.5** (Conlon–Zhao). Any \( n \)-vertex \( d \)-regular graph that is vertex transitive and \( \varepsilon \)-uniform is an \( (n, d, \lambda) \)-graph for \( \lambda \leq \Omega((\varepsilon d)) \).

Bilu and Linial [BL06] proved that if a \( d \)-regular graph satisfies the stronger condition that the left-hand side of (1) is at most \( \varepsilon d \sqrt{|S||T|} \), then \( \lambda \leq C \varepsilon d \log(2/\varepsilon) \) for some absolute constant \( C > 0 \). Theorem 1.5 gives the stronger conclusion \( \lambda \leq 4\varepsilon K G d \) from the weaker condition (1).

## 2. A link with Grothendieck’s inequality

The proof of Theorem 1.5 uses Grothendieck’s inequality, which we now recall.

**Theorem 2.1** (Grothendieck’s inequality). There exists an absolute constant \( K G \in (1, 2) \) such that the following holds. For any positive integer \( n \) and matrix \( B \in \mathbb{R}^{n \times n} \), we have

\[
\|B\| \leq K G \|B\|_{\infty \to 1}.
\]

For an \( n \)-vertex \( d \)-regular graph \( G \) with adjacency matrix \( A \), we use two simple propositions. Let \( J \) denotes the all-ones matrix.

**Proposition 2.2.** Let \( G \) be an \( n \)-vertex \( d \)-regular graph and let \( A \) be its adjacency matrix. Let \( B = A - \frac{d}{n} J \). Then,

\[
\lambda(G) = \|B\|
\]

\[
\|B\|_{\infty \to 1} \leq 4dn \Delta(G).
\]

The following key lemma allows us to apply Grothendieck’s inequality.

**Lemma 2.3.** Let \( G \) be a vertex-transitive \( n \)-vertex \( d \)-regular graph and let \( A \) be its adjacency matrix. Let \( B = A - \frac{d}{n} J \). Then,

\[
n\|B\| \leq \|B\|_G.
\]
Proof of Theorem 1.5: Let $A$ be the adjacency matrix of $G$ and let $B = A - \frac{d}{n}J$. Then, by Proposition 2.2 and Lemma 2.3,

$$n\lambda(G) = n\|B\| \leq \|B\|_G \leq K_G\|B\|_{\infty \to 1} \leq 4dnK_G\Delta(G).$$

\[\square\]

3. Proof of Lemma 2.3

The following proof of Lemma 2.3, which is even shorter than the original, follows from Grothendieck’s factorization lemma.

Lemma 3.1 (Grothendieck). For any matrix $A \in \mathbb{R}^{n \times n}$, there exist positive unit vectors $u, v \in \mathbb{R}^n_0 \cap S^{n-1}$ such that for any $x, y \in \mathbb{R}^n$,

$$|\langle Ax, y \rangle| \leq \|A\|_G \|x \circ u\|_2 \|y \circ v\|_2.$$  

For a permutation $\pi \in S_n$ and $A \in \mathbb{R}^{n \times n}$, let $A^\pi = (A_{\pi(i),\pi(j)})_{i,j=1}^n$. Observe that if $A$ is the adjacency matrix of a graph $G$ and $\pi$ is an automorphism of $G$, then $A^\pi = A$.

Proof of Lemma 2.3: Define $C = B/\|B\|_G$, so that $\|C\|_G = 1$. By Lemma 3.1 and the AMGM inequality, there exist positive unit vectors $u, v \in \mathbb{R}^n_0 \cap S^{n-1}$ such that for any $x, y \in \mathbb{R}^n$,

$$|\langle Cx, y \rangle| \leq \|x \circ u\|_2 \|y \circ v\|_2 \leq \frac{1}{2}(\|x \circ u\|_2^2 + \|y \circ v\|_2^2).$$

Fix $x, y \in \mathbb{R}^n$. Let $\Gamma \leq S_n$ be the group of automorphisms of $G$ and let $\Gamma$ act on $\mathbb{R}^n$ in the natural way. Since the adjacency matrix $A$ satisfies $A^\pi = A$ for every $\pi \in \Gamma$, it follows that $C^\pi = C$ for every $\pi \in \Gamma$, and therefore,

$$\langle Cx, y \rangle = \langle C^\pi x, y \rangle = \langle C(\pi^{-1}x), (\pi^{-1}y) \rangle.$$  

Putting the above two observations together gives

$$|\langle Cx, y \rangle| = |E_{\pi \in \Gamma}[\langle C(\pi^{-1}x), (\pi^{-1}y) \rangle]|$$

$$\leq \frac{1}{2}E_{\pi \in \Gamma}[\|((\pi^{-1}x) \circ u\|_2^2] + \frac{1}{2}E_{\pi \in \Gamma}[\|((\pi^{-1}y) \circ v\|_2^2]$$

$$= \frac{1}{2}E_{\pi \in \Gamma}[\|x \circ (\pi u)\|_2^2] + \frac{1}{2}E_{\pi \in \Gamma}[\|y \circ (\pi v)\|_2^2].$$
Since $\Gamma$ is transitive and $u$ is a unit vector, the first expectation on the last line equals
\[
E_{\pi \in \Gamma} \left[ \sum_{i=1}^{n} x_i^2 u_{\pi(i)}^2 \right] = \sum_{i=1}^{n} x_i^2 E_{\pi \in \Gamma} \left[ u_{\pi(i)}^2 \right] = n^{-1} \|x\|_2^2.
\]
Applying the same argument to the second expectation gives
\[
|\langle Cx, y \rangle| \leq \frac{1}{2n} (\|x\|_2^2 + \|y\|_2^2).
\]
Let $\lambda = \sqrt{\|y\|/\|x\|}$. Then, for $x' = \lambda x$ and $y' = \lambda^{-1} y$ we get that
\[
2|\langle Cx, y \rangle| = 2|\langle Cx', y' \rangle| \leq \frac{1}{n} (\|x'\|^2 + \|y'\|^2) = \frac{2}{n} \|x\| \|y\|.
\]
This shows that $\|C\| \leq 1/n$ and proves the claim. \qed

4. Exercises

Exercise 4.1. Prove Lemma 1.3 (the Expander mixing lemma).

Exercise 4.2. Prove Proposition 2.2.

Exercise 4.3. Show that in fact equality holds in (5). [Hint: use the fact that both the rows and the columns of $A - \frac{d}{n} J$ sum to zero.]

Exercise 4.4. Show that equality holds in Lemma 2.3.

References

