# LECTURE NOTES ON APPLICATIONS OF GROTHENDIECK'S INEQUALITY

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# NONLOCAL GAMES

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Abstract. In this lecture we cover one of the earliest application of Grothendieck's inequality, which appeared in the study of quantum entanglement. In particular, we will discuss Tsirelson's theorem [Tsi87] on XOR games.

## 1. Nonlocal games

Entanglement is arguably the most striking features of quantum mechanics. It is the possibility for two or more systems to be connected in a way that manifests itself when the systems are measured locally. Provided the systems were in an entangled state, the measurement results will be random and *correlated*. The correlations that entanglement can produce are special in the sense that they cannot be obtained with a source of randomness that is common to all the measurement results (referred to as shared randomness in computer science and local hidden variables in physics). For this reason, these correlations are often referred to as non-local.

Nonlocal games, introduced formally in [CHTW04], provide a useful model in which one can study the effects of entanglement. A two-player nonlocal game is defined by four finite sets $A, B, S, T$, a probability distribution $\pi : S \times T \to [0, 1]$ and a map $V : A \times B \times S \times T \to \{0, 1\}$. The map $V$ is usually referred to as the *predicate*. The game actually involves three parties, two players, usually called Alice and Bob, and a referee. The probability distribution and predicate are known to everyone. Before the game begins, Alice and Bob may come together to decide on a strategy to play the game. But after the game has begun, they are not allowed to communicate with each other anymore. At the start of the game, the referee picks a pair $(s, t) \in S \times T$ according to $\pi$, and sends $s$ to Alice and $t$ to Bob. Based on their strategy, the two players then answer the referee with $a \in A$ and $b \in B$, respectively. The players win the game if $V(a,b,s,t) = 1$, and lose otherwise. The players' objective is of course to maximize their probability of winning, where the probability is taken over the distribution $\pi$ and the possible randomness used in the players' strategy.

A strategy is captured completely by the joint conditional probability distribution $p[a,b \mid s,t]$ telling us the probability that questions $(s,t)$ are answered with $(a,b)$. Based on this, the winning probability is given by

$$\sum_{(s,t)\in S\times T} \pi_{st} \sum_{(a,b)\in A\times B} V(a,b,s,t)\, p[a,b \mid s,t].$$

A *deterministic classical strategy* is one in which the players determine in advance what answer to give to each question. That is to say that there are mappings $f : S \to A$ and $g : T \to B$ such that the question pair $(s,t)$ is answered with $(f(s), g(t))$. In a randomized classical strategy, the players may use private and shared randomness to decide their answers. The *classical value* of a game is the maximum winning probability under randomized classical strategies. It is not hard to see that the classical value can always be achieved with a deterministic strategy. In an *entangled strategy* the players decide what to answer based on the outcomes of local measurements on an entangled state.

## 2. Entangled strategies

To define what entangled strategies are we need to describe what quantum mechanics says about measurements on physical systems. In the following we endow $\mathbb{C}^d$ with the usual inner product $\langle x,y\rangle = \sum_{i=1}^d \overline{x_i} y_i$. Recall that a Hermitian matrix $M \in \mathbb{C}^{d\times d}$ is positive semidefinite if it has only nonnegative eigenvalues, or equivalently, if for any vector $x \in \mathbb{C}^d$, we have that $\langle Mx, x\rangle \geq 0$.

2.1. **Quantum mechanical systems.** A first postulate is that a physical system $\mathsf{X}$ is represented by a complex vector space $\mathcal{X}$ with an inner product, making $\mathcal{X}$ a Hilbert space. In our setting, we shall only deal with finite dimensions and have $\mathcal{X} = \mathbb{C}^d$ for some dimension $d$. The set of possible states of $\mathsf{X}$ is given by the set of unit vectors in $\mathcal{X}$.

2.2. **Measurements.** A *measurement* on a system $\mathsf{X}$ represented by a vector space $\mathcal{X} = \mathbb{C}^d$ is given by a set $\Gamma$ and a collection of $d \times d$

positive semidefinite matrices $\mathcal{M} = \{M_a : a \in \Gamma\}$ with the property

$$\sum_{a\in\Gamma} M_a = I_d,$$

where $I_d$ is the identity matrix in $\mathbb{C}^{d\times d}$. The set $\Gamma$ corresponds to the possible measurement outcomes. A second postulate is that if $\mathsf{X}$ is in the state $\psi \in \mathcal{X}$ and the measurement $\mathcal{M}$ is performed on $\mathsf{X}$, then the probability of obtaining outcome $a \in \Gamma$ is given by $\langle M_a\psi, \psi\rangle$.

2.3. **Composite systems.** A third postulate is that if $\mathsf{X}, \mathsf{Y}$ are two physical systems, then the set of possible states of their union is given by the unit vectors in the *tensor product* of their representing vector spaces

$$\mathcal{X} \otimes \mathcal{Y} = \operatorname{Span}\big\{x \otimes y : x \in \mathcal{X},\ y \in \mathcal{Y}\big\}.$$

A state of the composite system $(\mathsf{X}, \mathsf{Y})$ is said to be *entangled* if it is *not* of the form $x \otimes y$ for some unit vectors $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. A basic famous example of an entangled state is the so-called EPR pair $(e_0 \otimes e_0 + e_1 \otimes e_1)/\sqrt{2}$, where $e_0, e_1 \in \mathbb{C}^2$ are the standard basis vectors.

2.4. **Local measurements on composite systems.** Consider a composite system $(\mathsf{X}, \mathsf{Y})$ in some state $\psi \in \mathcal{X} \otimes \mathcal{Y}$. Let $\mathcal{A} = \{A_a : a \in \Gamma\}$ and $\mathcal{B} = \{B_b : b \in \Lambda\}$ be measurements on $\mathsf{X}, \mathsf{Y}$, respectively. A fourth and final postulate is that if $\mathcal{A}$ is performed on $\mathsf{X}$ and $\mathcal{B}$ is performed on $\mathsf{Y}$, then then the probability that the measurements yield outcomes $a \in \Gamma, b \in \Lambda$, respectively, is given by $\langle A_a \otimes B_b\psi, \psi\rangle$.

2.5. **Entangled strategies.** An entangled strategy for a game, then, corresponds to a pair of physical systems $(\mathsf{X}, \mathsf{Y})$ prepared in some state $\psi \in \mathcal{X} \otimes \mathcal{Y}$, an $A$-valued measurement $\mathcal{A}^s$ for each $s \in S$ and a $B$-valued measurement $\mathcal{B}^t$ for each $t \in T$. Alice will hold system $\mathsf{X}$ and Bob will hold system $\mathsf{Y}$. To answer the questions $s \in S$ and $t \in T$, the players will measure their respective systems with measurements $\mathcal{A}^s$ and $\mathcal{B}^t$, respectively. The probability that they answer with $(a, b)$ is then given by

$$p(a, b \mid s, t) = \langle A^s_a \otimes B^t_b\psi, \psi\rangle.$$

## 3. The CHSH game

There are many examples of nonlocal games for which entangled strategies can beat the best classical strategy, in the sense that the entangled value is strictly larger than the classical value. The simplest of these games is the CHSH game. Here $A = B = S = T = \mathbb{Z}_2$ and the players win if and only if the sum of their answers equals the product of their questions, and questions are picked uniformly at random. A perfect classical strategy requires a solution to the system of linear equations

$$\begin{aligned} a_0 + b_0 &= 0 \\ a_0 + b_1 &= 0 \\ a_1 + b_0 &= 0 \\ a_1 + b_1 &= 1. \end{aligned}$$

Since the sum (modulo 2) of the left- and right-hand sides of these equations are not equal, we see that the classical value is at most 3/4. Clearly this can be attained if the players always answer 0.

Surprisingly, the entangled value of this game is $\cos(\pi/8)^2 \approx 0.85$! Instead of describing the strategy that does this, we will move on to a celebrated result of Tsirelson's, from which this can easily be deduced. To this end, we place the CHSH game in a general class of nonlocal games known as XOR games.

## 4. XOR games

An XOR game is given by a matrix $M \in \{-1, 1\}^{n \times n}$ and a probability distribution $\pi$ over its coordinates $[n] \times [n]$. In such a game, the referee samples a matrix coordinate $(s, t)$ according to $\pi$, sends $s$ to Alice, $t$ to Bob, so $S = T = [n]$. The players then each answer with a sign, so $A = B = \{-1, 1\}$ and win if and only if the product of their answers equals $M_{st}$. It is not hard to see that CHSH corresponds to the game with game matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$.

Notice that random play ensures that any XOR games can trivially be won with probability at least $\frac{1}{2}$. For this reason, the typical figure of merit in XOR games is the *bias*, twice the amount by which the winning probability can exceed $\frac{1}{2}$. Equivalently, the bias is the probability of winning minus the probability of losing. For an XOR game $\mathcal{G}$, denote by $\beta(\mathcal{G})$ and $\beta^*(\mathcal{G})$ the maximum biases under classical and entangled strategies.

**Proposition 4.1.** *Let $\mathcal{G}$ be an XOR game given by a game matrix $M \in \{-1, 1\}^{n \times n}$ and probability distribution $\pi$. Then, $\beta(\mathcal{G}) = \|M \circ \pi\|_{\infty \to 1}$, where $M \circ \pi = (M_{ij}\pi_{ij})_{i,j=1}^n$.*

*Proof:* Let $f, g : [n] \to \{-1,1\}^n$ be some deterministic classical strategies for $\mathcal{G}$. Notice that $M_{ij} f(i) g(j) = +1$ if the players win on $(i,j)$ and $-1$ otherwise. Therefore, the bias can be expressed as

$$\sum_{i,j=1}^{n} \pi_{ij} M_{ij} f(i) g(j),$$

which is clearly at most $\|M\|_{\infty \to 1}$. □

## 5. Tsirelson's theorem

Tsirelson proved that the entangled bias of any XOR game is given in terms of the Grothendieck norm of the matrix $M \circ \pi = (M_{st}\pi_{st})_{s,t=1}^{n}$.

**Theorem 5.1** (Tsirelson). *Let $\mathcal{G}$ be an XOR game with game matrix $M \in \{-1,1\}^{n \times n}$ and probability distribution $\pi : [n] \times [n] \to [0,1]$. Then, $\beta^*(\mathcal{G}) = \|M \circ \pi\|_G$.*

Combining this with Proposition 4.1 and Grothendieck's inequality, we get the following immediate corollary.

**Corollary 5.2.** *For a any XOR game $\mathcal{G}$, we have $\beta^*(\mathcal{G}) \leq K_G \beta(\mathcal{G})$.*

A key ingredient for Theorem 5.1 is the following lemma.

**Lemma 5.3.** *For each $d \in \mathbb{N}$ there is a $D \in \mathbb{N}$ and symmetric matrices $C_1, \dots, C_d \in \mathbb{R}^{D \times D}$ such that $C_i^2 = I_N$ and $C_i C_j = -C_j C_i$ for all $i \neq j$.*

*Proof:* Define the following three matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Observe that each of these is symmetric and squares to $I$. For each $i \in [d]$, define

$$C_i = \underbrace{I \otimes \cdots \otimes I}_{d-i} \otimes X \otimes \underbrace{Z \otimes \cdots \otimes Z}_{i-1}$$

Clearly these matrices are symmetric and square to the identity. The other desired property is easily verified using the fact $XZ = -ZX$. □

Since any real symmetric matrix that squares to the identity has only $\{-1,1\}$ eigenvalues, it is the difference of two (real) positive semidefinite matrices. From this, we get the following simple corollary of Lemma 5.3

**Corollary 5.4.** *For $x \in \mathbb{R}^d$, let $C(x) = x_1C_1 + \cdots + x_dC_d$. Then, for any $x, y \in \mathbb{R}^d$,*

$$\mathsf{Tr}(C(x)C(y)) = D\langle x, y\rangle$$

*If $x \in S^{d-1}$, then there exist real positive semidefinite matrices $C(x)_+, C(x)_-$ such that $C(x) = C(x)_+ - C(x)_-$, where $\{C(x)_+, C(x)_-\}$ is a measurement.*

*Proof of Theorem 5.1:* We first show that $\beta^*(\mathcal{G}) \leq \|M \circ \pi\|_G$. Let $\psi \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ be a unit vector and for each $s, t \in [n]$, let $\{A^s_+, A^s_-\}$ and $\{B^t_+, B^t_-\}$ be $\{-1, 1\}$-valued measurements on $\mathbb{C}^{d_1}$ and $\mathbb{C}^{d_2}$, respectively. For each $s, t \in [n]$ define $A^s = A^s_+ - A^s_-$ and $B^t = B^t_+ - B^t_-$. Then, the bias based on this strategy equals

$$\sum_{s,t=1}^{n} \pi_{st} M_{st} \Big( \sum_{a,b\in\{-1,1\}} ab \,\langle A^s_a \otimes B^t_b \psi, \psi\rangle \Big) = \sum_{s,t=1}^{n} \pi_{st} M_{st} \,\langle A^s \otimes B^t \psi, \psi\rangle.$$

Define the vectors

$$x_s = A^s \otimes I_{d_2}\psi \quad \text{and} \quad y_t = I_{d_1} \otimes B^t\psi.$$

and

$$u_s = \begin{pmatrix} \Re(x_s) \\ \Im(x_s) \end{pmatrix} \quad \text{and} \quad v_t = \begin{pmatrix} \Re(y_t) \\ -\Im(y_t) \end{pmatrix}.$$

Then, since $A^s$ and $B^t$ are Hermitian,

$$\langle u_s, v_t\rangle = \Re\big(\langle x_s, y_t\rangle\big) = \Re\big(\langle A^s \otimes B^t\psi, \psi\rangle\big) = \langle A^s \otimes B^t\psi, \psi\rangle.$$

We claim that $u_s, v_t$ have norm at most 1. First observe they their norm is equal to that of $x_s, y_t$, respectively. Moreover, it suffices to show that $A^s$ and $B^t$ have operator norm at most 1. We show this for $A^s$, the same argument being applicable to $B^t$. Since $A^s$ is Hermitian, it is enough to show that $|\langle A^s w, w\rangle| \leq 1$ for any unit vector $w$. To this end, observe that $A^s = 2A^s_+ - I_{d_1}$ and that $0 \leq \langle A^s_+ w, w\rangle \leq 1$, since $A^s_+ + A^s_- = I_{d_1}$ and $A^s_+$ is positive semidefinite. Hence, $\langle A^s w, w\rangle = 2\langle A^s_+ w, w\rangle - \langle w, w\rangle \in [-1, 1]$, which implies the claim.

Now we show that $\|M\|_G \leq \beta^*(\mathcal{G})$. For each $s \in S$ and $t \in T$, let $x_s, y_t \in S^{2n-1}$ be some unit vectors. Let $C : \mathbb{R}^{2n} \to \mathbb{C}^{D\times D}$ be the map as in Corollary 5.4 for $d = 2n$. Let

$$\psi = \frac{1}{\sqrt{D}} \sum_{i=1}^{D} e_i \otimes e_i,$$

where $e_i$ is the $i$th standard basis vector in $\mathbb{C}^D$. Clearly $\psi$ is a unit vector and so represents a valid state of a bi-partite quantum system composed of two $D$-dimensional parts. Observe that for any two matrices $A, B \in \mathbb{C}^{D\times D}$, we have

$$\langle A \otimes B\psi, \psi\rangle = \frac{1}{D}\sum_{i,j=1}^{D} A_{ij}B_{ij} = \mathsf{Tr}(A^{\mathsf{T}}B).$$

For each $s \in S$ and $t \in T$, define the measurements $\mathcal{A}^s = \{C(x_s)_+, C(x_s)_-\}$ and $\mathcal{B}^t = \{C(y_t)_+, C(y_t)_-\}$. If the players perform these measurements on their share of the state $\psi$, then the expected product of their measurement outcomes is easily seen to equal

$$\langle C(x_s) \otimes C(y_t)\psi, \psi\rangle = \mathsf{Tr}(C(x_s)C(y_t)) = \langle x_s, y_t\rangle.$$

Hence, the bias based on this strategy equals

$$\sum_{(s,t)\in S\times T} \pi_{st}M_{st}\langle x_s, y_t\rangle.$$

Taking a supremum over unit vectors $x_s, y_t$, we see that we can achieve a bias of $\|M \circ \pi\|_G$. □

## References

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